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ORIGINAL PAPER

On the rate of the convergence of the characteristic values of an integral operator associated with a dissipative fourth order differential operator in lim-4 case with finite transmission conditions

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Abstract In this paper, our main aim is to investigate the spectral properties of a singular dissipative fourth order boundary value problem in lim-4 case with finite transmission conditions. For this purpose we construct a suitable differential operator in an appropriate Hilbert space. After showing that this differential operator is a dissipative operator we pass to the resolvent operator with an explicit form. Using this resolvent operator and Krein's theorem we prove a completeness theorem on the boundary value transmission problem.

Keywords Dissipative operator · Fourth order operator · Transmission conditions · Completeness theorem

Mathematics Subject Classification 34B05 · 34B20 · 34B27 · 34B40

1 Introduction

Boundary value problems generated by the ordinary differential equations and appropriate boundary conditions are important to understand many real world problems. For example, oxygen diffusion in cells, heat and mass transfer within porous catalyst particle, astrophysics, the study of stellor interiors, flow networks in biology, control and optimization theory are closely related with boundary value problems [1–5]. In particular, fourth order boundary value problems arise in the study of mathematical

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modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [6–9] and therefore have attracted much attention in recent years.

As is known that differential equations can be considered on a single interval as well as on multi-interval. In the secondary case, differential equations may be handled with additional transmission conditions. By transmission conditions it is meant that the solution of a differential equation satisfies additional conditions at an inner point of the interval. In this case, differential equation may be handled on multi-interval. Such boundary value transmission problems have many applications in scientific problems. To be more precise we should note that many physical, chemical, biological phenomena involving thresholds, bursting rhythm models in medicine, pharmacokinetics and frequency modulated systems do exhibit transmission effects [10]. Therefore, the theory of differential operators generated by a differential expression and boundary value transmission conditions is a new and important branch of operator theory which has an extensive physical, chemical and realistic mathematical model and has been emerging as an important area of investigation.

In this paper we concern with the following fourth order differential equation

$$\varphi^{(4)} - \left(q_1(x)\varphi^{(1)}\right)^{(1)} + q_0(x)\varphi = \mu\varphi, \tag{1.1}$$

defined on the union of the intervals $I_k = (c_{k-1}, c_k)$ as $I = \bigcup_{k=1}^{n+1} I_k$. Here $\varphi^{(r)}$ denotes the ordinary r-th derivative of φ and μ is a complex parameter, Basic assumptions on the Eq. (1.1) and the intervals I_k are as follows:

(i) $-\infty < c_0 < c_1 < \ldots < c_{n+1} \leq \infty$,
(ii) $c_m, m = \overline{0, n} := 0, 1, \ldots, n$, are the regular points and c_{n+1} is the singular point for (1.1) and,
(iii) q_0 and q_1 are real-valued, Lebesgue measurable and locally integrable functions on all $I_k, k = \overline{1, n+1}$.

We shall impose some boundary and transmission conditions at the end points of the intervals $I_k, k = \overline{1, n+1}$. Then we will use the operator theory. However, for this purpose we shall construct a Hilbert space. Let $L^2(I)$ be the Hilbert space consisting of all squarely integrable functions φ such as

$$\int_I |\varphi|^2 \, dx < \infty$$

and equipped with the usual inner product

$$(\varphi, \chi) = \int_I \varphi\overline{\chi} dx.$$

Differential operators generated by the differential expressions and boundary value transmission conditions may be selfadjoint or nonselfadjoint in some Hilbert spaces. In particular, if the imaginary part of a nonselfadjoint operator acting on a Hilbert space is nonnegative, then the operator is called dissipative. A direct result is that all eigenvalues of a dissipative operator belong to the closed upper half-plane. However, the spectral analysis of a dissipative operator needs to be completed. An important problem is to describe the completeness of the root vectors (eigen- and associated vectors) of a dissipative operator. This completeness result may be used in some applications. For example, non-classical wavelets can be obtained from root vectors for nonselfadjoint problems [11]. In the literature there are some theorems for getting complete information for a dissipative operator. For example, Krein's theorem is one of the main theorems. It is better to note that some second order differential operators have been investigated by Krein's theorem [12,13]. On the other hand, in 2014 Zhang and Sun have studied a singular fourth order dissipative operator with a transmission point with the help of Livšic's theorem [14]. In this paper using Krein's theorem we investigate a singular dissipative fourth order differential operator generated by (1.1) and finite transmission conditions.

2 Basic solutions of the fourth order equation

In this section we shall introduce some basic solutions of the Eq. (1.1). However, at first we shall introduce the quasi-derivatives $\varphi^{[r]}$ of φ as follows (see [15])

$$\begin{aligned}
&\varphi^{[0]} = \varphi,\\
&\varphi^{[1]} = \varphi^{(1)},\\
&\varphi^{[2]} = \varphi^{(2)},\\
&\varphi^{[3]} = q_1\varphi^{(1)} - \varphi^{(3)},\\
&\varphi^{[4]} = q_0\varphi - (q_1\varphi^{(1)})^{(1)} + \varphi^{(4)}.
\end{aligned}$$

In this case (1.1) can be rewritten as

$$\varphi^{[4]} = \mu\varphi, \quad x \in I.$$

Green's formula may help us to impose the boundary condition at the singular point. Therefore we shall describe a suitable set. Let D be a set in $L^2(I)$ consisting of all functions $\varphi \in L^2(I)$ satisfying $\varphi^{[r-1]}$, $r = \overline{1,4}$, are locally absolutely continuous on all I_k, $k = \overline{1, n+1}$, and $\varphi^{[4]} \in L^2(I)$. Hence for all $\varphi, \chi \in D$, we obtain on I_k that

$$\int\limits_{I_k} \left\{\varphi^{[4]}\chi - \varphi\chi^{[4]}\right\} dx = [\varphi, \chi]_{c_{k-1}+}^{c_k-}, \tag{2.1}$$

where $[\varphi, \chi]_{c_{k-1}+}^{c_k-} = [\varphi, \chi](c_k-) - [\varphi, \chi](c_{k-1}+)$ and

$$[\varphi, \chi] = \varphi^{[0]}\chi^{[3]} - \varphi^{[3]}\chi^{[0]} + \varphi^{[1]}\chi^{[2]} - \varphi^{[2]}\chi^{[1]}. \tag{2.2}$$

Equation (2.2) is called the Lagrange form of the Eq. (1.1) and is equivalent to the following

$$[\varphi, \chi] = q_1\left(\varphi\chi^{(1)} - \varphi^{(1)}\chi\right) + \left(\varphi^{(3)}\chi - \varphi^{(2)}\chi^{(1)} + \varphi^{(1)}\chi^{(2)} - \varphi\chi^{(3)}\right). \tag{2.3}$$

In the papers [16] and [17], Everitt called the Eq. (2.3) as bilinear concomitant of φ and χ. It is clear from (2.1) and (2.2) [equivalently (2.3)] that if $\varphi(x,\mu)$ and $\chi(x,\mu)$ are the solutions of (1.1) for the same value of μ, then $[\varphi,\chi]$ is independent of x and depends only on μ on each interval I_k, $k=\overline{1,n+1}$. Hence we have the following Green's formula

$$\int_I \left\{\varphi^{[4]}\chi - \varphi\chi^{[4]}\right\} dx = \sum_{k=1}^{n+1} [\varphi,\chi]_{c_{k-1}+}^{c_k-}. \tag{2.4}$$

In particular, (2.1) or (2.4) implies that for arbitrary $\varphi,\chi \in D$, at singular point c_{n+1}, the values $[\varphi,\chi](c_{n+1}-)$ and $[\varphi,\overline{\chi}](c_{n+1}-)$ exist and are finite. Latter one also follows from Green's formula (2.4) [or (2.1)]. In fact, it is sufficient to get the second factor with its complex conjugate.

One of the useful tools for studying the spectral properties of a singular differential operator is Weyl's limit-point/circle theory. In fact, in 1910 Weyl proved that the nested circles of the corresponding regular second order boundary value problems converge either to a circle or a point in the corresponding m-plane [18]. These results imply that at least one of the linearly independent solutions of a second order singular differential equation defined on a semi-infinite interval must be squarely integrable. However, two linearly independent solutons and combinations of them may be squarely integrable. While primary case is known as limit-point case, secondary case is known as limit-circle case for a second order differential operator. In 1946, Titchmarsh introduced some results in the regular second order case and developted Weyl's limit-point/circle theory as introducing some properties of the corresponding m-functions [19]. In 1963, Everitt constructed Weyl's theory for the singular fourth order differential operator [17] and using the connection between the dimension of the limit surface and the number of squarely integrable solutions of a fourth order differential equation, Everitt proved that at least two linearly independent solutions of a singular fourth order differential equation must be squarely integrable on some semi-infinite intervals. Moreover, three or four linearly independent solutions can be squarely integrable. These cases are called lim-2, lim-3 and lim-4 cases, respectively, for the fourth order case. In particular, lim-2 case is known as Weyl's limit-point case and lim-4 case is known as limit-circle case for the fourth order differential operator.

We assume that lim-4 case holds for the Eq. (1.1) (see [20–24]).

Let

$$\varphi_r(x,\mu) = \begin{cases} \varphi_{r1}(x,\mu), & x \in I_1 \\ \varphi_{r2}(x,\mu), & x \in I_2 \\ \vdots & \\ \varphi_{r(n+1)}(x,\mu), & x \in I_{n+1} \end{cases},$$

where $r=\overline{1,4}$, be the solutions of $\varphi^{[4]} = \mu\varphi$, $\varphi \in D$, $x \in I$. We use the notation

$$W_x(\varphi_{1k},\ldots,\varphi_{jk})(\mu) = \det \begin{bmatrix} \varphi_{1k}(x,\mu) & \cdots & \varphi_{jk}(x,\mu) \\ \vdots & & \vdots \\ \varphi_{1k}^{(j-1)}(x,\mu) & \cdots & \varphi_{jk}^{(j-1)}(x,\mu) \end{bmatrix}, \quad x \in I_k,$$

where $k = \overline{1, n+1}, 1 \le j \le 4$, to denote the Wronskian of the set $\{\varphi_{rk}(x,\mu) : 1 \le r \le j\}$ of order j on each interval I_k. For $j = 4$, it is known that the equality

$$W_x(\varphi_{1k}, \ldots, \varphi_{4k})(\mu) = -[\varphi_{1k}, \varphi_{2k}](x,\mu)[\varphi_{3k}, \varphi_{4k}](x,\mu) + [\varphi_{1k}, \varphi_{3k}](x,\mu)[\varphi_{2k}, \varphi_{4k}](x,\mu) - [\varphi_{1k}, \varphi_{4k}](x,\mu)[\varphi_{2k}, \varphi_{3k}](x,\mu), \tag{2.5}$$

holds on each I_k, $k = \overline{1, n+1}$ (see [16]). Equation (2.5) implies that the Wronskian of $\varphi_{1k}(x,\mu), \ldots, \varphi_{4k}(x,\mu)$ is independent of x and depends only on μ on each I_k, $k = \overline{1, n+1}$.

Now we shall describe the basic solutions of $\varphi^{[4]} = \mu\varphi$, $x \in I$. Let

$$\theta_m(x,\mu) = \begin{cases} \theta_{m1}(x,\mu), x \in I_1 \\ \theta_{m2}(x,\mu), x \in I_2 \\ \vdots \\ \theta_{m(n+1)}(x,\mu), x \in I_{n+1} \end{cases}, \quad \psi_m(x,\mu) = \begin{cases} \psi_{m1}(x,\mu), x \in I_1 \\ \psi_{m2}(x,\mu), x \in I_2 \\ \vdots \\ \psi_{m(n+1)}(x,\mu), x \in I_{n+1} \end{cases},$$

where $m = 1, 2$, be the solutions of (1.1) satisfying the conditions

$$\begin{array}{llll} \theta_{11}^{[0]}(c_0+,\mu) = \alpha_1, & \theta_{11}^{[1]}(c_0+,\mu) = 0, & \theta_{11}^{[2]}(c_0+,\mu) = 0, & \theta_{11}^{[3]}(c_0+,\mu) = 1, \\ \theta_{21}^{[0]}(c_0+,\mu) = 0, & \theta_{21}^{[1]}(c_0+,\mu) = \alpha_2, & \theta_{21}^{[2]}(c_0+,\mu) = 1, & \theta_{21}^{[3]}(c_0+,\mu) = 0, \\ \psi_{11}^{[0]}(c_0+,\mu) = \beta_1, & \psi_{11}^{[1]}(c_0+,\mu) = 0, & \psi_{11}^{[2]}(c_0+,\mu) = 0, & \psi_{11}^{[3]}(c_0+,\mu) = 1, \\ \psi_{21}^{[0]}(c_0+,\mu) = 0, & \psi_{21}^{[1]}(c_0+,\mu) = \beta_2, & \psi_{21}^{[2]}(c_0+,\mu) = 1, & \psi_{21}^{[3]}(c_0+,\mu) = 0, \end{array}$$

and

$$\begin{array}{ll} \theta_{m(s+1)}^{[0]}(c_s+,\mu) = \frac{1}{\gamma_{1s}}\theta_{ms}^{[0]}(c_s-,\mu), & \psi_{m(s+1)}^{[0]}(c_s+,\mu) = \frac{1}{\gamma_{1s}}\psi_{ms}^{[0]}(c_s-,\mu), \\ \theta_{m(s+1)}^{[1]}(c_s+,\mu) = \frac{1}{\gamma_{2s}}\theta_{ms}^{[1]}(c_s-,\mu), & \psi_{m(s+1)}^{[1]}(c_s+,\mu) = \frac{1}{\gamma_{2s}}\psi_{ms}^{[1]}(c_s-,\mu), \\ \theta_{m(s+1)}^{[2]}(c_s+,\mu) = \frac{1}{\gamma_{3s}}\theta_{ms}^{[2]}(c_s-,\mu), & \psi_{m(s+1)}^{[2]}(c_s+,\mu) = \frac{1}{\gamma_{3s}}\psi_{ms}^{[2]}(c_s-,\mu), \\ \theta_{m(s+1)}^{[3]}(c_s+,\mu) = \frac{1}{\gamma_{4s}}\theta_{ms}^{[3]}(c_s-,\mu), & \psi_{m(s+1)}^{[3]}(c_s+,\mu) = \frac{1}{\gamma_{4s}}\psi_{ms}^{[3]}(c_s-,\mu), \end{array}$$

where $m = 1, 2$, $s = \overline{1, n}$, α_m, β_m and γ_{js} are real numbers satisfying $\alpha_1 - \beta_1 = 1$, $\alpha_2 - \beta_2 = 1$ and

$$\Upsilon_{(s)} := \gamma_{1s}\gamma_{4s} = \gamma_{2s}\gamma_{3s} > 0.$$

Clearly the equalities

$$\begin{array}{lll} [\theta_{r1}, \psi_{m1}] = \delta_{rm}, & [\theta_{r1}, \theta_{m1}] = 0, & [\psi_{r1}, \psi_{m1}] = 0, \\ [\theta_{r2}, \psi_{m2}] = \left(\Upsilon_{(1)}\right)^{-1}\delta_{rm}, & [\theta_{r2}, \theta_{m2}] = 0, & [\psi_{r2}, \psi_{m2}] = 0, \\ \vdots & \vdots & \vdots \\ [\theta_{r(n+1)}, \psi_{m(n+1)}] = \left(\prod_{k=1}^{n}\Upsilon_{(k)}\right)^{-1}\delta_{rm}, & [\theta_{r(n+1)}, \theta_{m(n+1)}] = 0, & [\psi_{r(n+1)}, \psi_{m(n+1)}] = 0, \end{array} \tag{2.6}$$

where $1 \leq r, m \leq 2$ and δ_{rm} is the Kronecker delta, hold.

Let $u_m(x) = \theta_m(x, 0)(x \in I)$, $z_m(x) = \psi_m(x, 0)(x \in I)$, where $m = 1, 2$,

$$u_m(x) = \begin{cases} u_{m1}(x), x \in I_1 \\ u_{m2}(x), x \in I_2 \\ \vdots \\ u_{m(n+1)}(x), x \in I_{n+1} \end{cases}, \quad z_m(x) = \begin{cases} z_{m1}(x), x \in I_1 \\ z_{m2}(x), x \in I_2 \\ \vdots \\ z_{m(n+1)}(x), x \in I_{n+1} \end{cases},$$

$u_{mk}(x) = \theta_{mk}(x, 0)(x \in I_k)$ and $z_{mk}(x) = \psi_{mk}(x, 0)(x \in I_k)$, $k = \overline{1, n+1}$. Then one can say that $\{u_1, u_2, z_1, z_2\}$ are the real solutions of $\varphi^{[4]} = 0$, $x \in I$.

It is better to note that since lim-4 case holds for (1.1), the solutions θ_m, ψ_m, u_m, z_m, where $m = 1, 2$, and $j = \overline{1, 4}$, belong to $L^2(I)$ and D. Therefore for arbitrary $\varphi \in D$ the values $[\varphi, z_m](c_{n+1}-)$ and $[\varphi, u_m](c_{n+1}-)$ exist and are finite.

Let us define $\omega_k(\mu) := W_x(\theta_{1k}, \theta_{2k}, \psi_{1k}, \psi_{2k})(\mu)$ on each I_k, $k = \overline{1, n+1}$. Using (2.5) and (2.6) it is easy to see that

$$\omega_1(\mu) = 1, \quad \omega_2(\mu) = \left(\Upsilon_{(1)}\right)^{-2}, \cdots, \quad \omega_{n+1}(\mu) = \left(\prod_{r=1}^{n} \Upsilon_{(r)}\right)^{-2}.$$

Let $\omega_k := \omega_k(0)$. Then it is clear that $\omega_k = W_x(u_{1k}, u_{2k}, z_{1k}, z_{2k})$, $x \in I_k$, and

$$\omega_1 = 1, \quad \omega_2 = \left(\Upsilon_{(1)}\right)^{-2}, \cdots, \quad \omega_{n+1} = \left(\prod_{r=1}^{n} \Upsilon_{(r)}\right)^{-2}.$$

3 Some identities and entire functions

Plücker's identity is useful to study the singular problems and in the literature it is obtained some identities on single intervals. However, it is necessary for us to obtain some identities on multi-interval. For this purpose we use Fulton's idea [25].

Let us consider the following association

$$y \longleftrightarrow Y = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{bmatrix}. \tag{3.1}$$

Then we associate all u_{ms} and z_{ms} with U_{ms} and Z_{ms}, respectively, as follows $u_{ms} \longleftrightarrow U_{ms}$ and $z_{ms} \longleftrightarrow Z_{ms}$, where $m = 1, 2$, $s = \overline{1, n+1}$. We construct 4×4 matrices on each I_s as

$$A_s = [U_{1s}, U_{2s}, Z_{1s}, Z_{2s}]$$

and let

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix},$$

where I denotes the 2×2 unit matrix. It is clear that $[y, w] = W^t JY$, where $w \longleftrightarrow W$ and W^t denotes the transpose of the matrix W. With a direct calculation we arrive at

$$A_1^t J A_1 = J \tag{3.2}$$

and for $k = \overline{2, n+1}$

$$A_k^t J A_k = \left(\prod_{r=1}^{k-1} \Upsilon_{(r)} \right)^{-1} J. \tag{3.3}$$

Define the transformation

$$\mathcal{S} Y_s = A_s^{-1} Y_s, \tag{3.4}$$

where Y_s is associated with the solution y of (1.1) under (3.1) on I_s, $s = \overline{1, n+1}$. Since $A_s.(\mathcal{S}Y_s) = Y_s$, we obtain from Cramer's rule that

$$(\mathcal{S}Y_1)(x) = \begin{bmatrix} [y_1, z_{11}](x) \\ [y_1, z_{21}](x) \\ -[y_1, u_{11}](x) \\ -[y_1, u_{21}](x) \end{bmatrix}, \ (\mathcal{S}Y_k)(x) = \left(\prod_{r=1}^{k-1} \Upsilon_{(r)} \right) \begin{bmatrix} [y_k, z_{1k}](x) \\ [y_k, z_{2k}](x) \\ -[y_k, u_{1k}](x) \\ -[y_k, u_{2k}](x) \end{bmatrix}. \tag{3.5}$$

Using (3.4) and (3.2) we find that

$$(\mathcal{S}W_1)^t J (\mathcal{S}Y_1) = W_1^t J Y_1. \tag{3.6}$$

Similarly from (3.4) and (3.3) we obtain for $k = \overline{2, n+1}$ that

$$(\mathcal{S}W_k)^t J (\mathcal{S}Y_k) = \left(\prod_{r=1}^{k-1} \Upsilon_{(r)} \right) W_k^t J Y_k. \tag{3.7}$$

Therefore using (3.5)–(3.7) we obtain for

$$\varphi(x) = \begin{cases} \varphi_1(x), x \in I_1 \\ \varphi_2(x), x \in I_2 \\ \vdots \\ \varphi_{n+1}(x), x \in I_{n+1} \end{cases}, \ \chi(x) = \begin{cases} \chi_1(x), x \in I_1 \\ \chi_2(x), x \in I_2 \\ \vdots \\ \chi_{n+1}(x), x \in I_{n+1} \end{cases} \in D$$

that

$$\begin{aligned} [\varphi_1, \chi_1] = {} & [\varphi_1, u_{11}][\chi_1, z_{11}] - [\varphi_1, z_{11}][\chi_1, u_{11}] + [\varphi_1, u_{21}][\chi_1, z_{21}] \\ & -[\varphi_1, z_{21}][\chi_1, u_{21}], \quad x \in I_1 \\ [y_2, \chi_2] = {} & \Upsilon_{(1)} \{ [\varphi_2, u_{12}][\chi_2, z_{12}] - [\varphi_2, z_{12}][\chi_2, u_{12}] + [\varphi_2, u_{21}][\chi_2, z_{22}] \\ & -[\varphi_2, z_{22}][\chi_2, u_{22}] \}, \quad x \in I_2 \\ \vdots & \end{aligned}$$

$$
\begin{aligned}
[\varphi_{n+1}, \chi_{n+1}] = \prod_{r=1}^{n} \Upsilon_{(r)} \big\{ & [\varphi_{n+1}, u_{1(n+1)}][\chi_{n+1}, z_{1(n+1)}] \\
& -[\varphi_{n+1}, z_{1(n+1)}][\chi_{n+1}, u_{1(n+1)}] + [\varphi_{n+1}, u_{2(n+1)}][\chi_{n+1}, z_{2(n+1)}] \\
& -[\varphi_{n+1}, z_{2(n+1)}][\chi_{n+1}, u_{2(n+1)}] \big\}, \ x \in I_{n+1}.
\end{aligned} \tag{3.8}
$$

With this identities we describe the growth of some entire functions. Therefore it is better to remind some definition and results.

An entire function $g(\mu)$ is called of order≤ 1 and minimal type if for each $\epsilon > 0$ the following inequality holds [26]

$$|g(\mu)| \leq D_\epsilon e^{\epsilon|\mu|}, \quad \mu \in \mathbb{C}, \tag{3.9}$$

where D_ϵ is a constant. If an entire function $g(\mu)$ satisfies the inequality (3.9) for each $\epsilon > 0$, then

$$\limsup_{|\mu|\to\infty} \frac{1}{|\mu|} \log|g(\mu)| \leq 0. \tag{3.10}$$

It is known that [27] if an entire function $g(\mu)$ satisfies (3.10) and $g(0) = 1$, then $g(\mu)$ has the representation

$$g(\mu) = \lim_{r\to\infty} \prod_{|\mu_j|\leq r} \left(1 - \frac{\mu}{\mu_j}\right),$$

and the limit $\lim_{r\to\infty} \sum_{|\mu_j|\leq r} 1/\mu_j$ exists and is finite. Then we have the following theorem.

Theorem 3.1 *The functions* $[\theta_r(x,\mu), z_m(x)](c_{n+1}-)$, $[\theta_r(x,\mu), u_m(x)](c_{n+1}-)$, $[\psi_r(x,\mu), z_m(x)](c_{n+1}-)$ *and* $[\psi_r(x,\mu), u_m(x)](c_{n+1}-)$, *where* $1 \leq r, m \leq 2$, *are entire functions of* μ *of order* ≤ 1 *and are of minimal type.*

Proof Consider the solution

$$\varphi(x,\mu) = \begin{cases} \varphi_1(x,\mu), x \in I_1 \\ \varphi_2(x,\mu), x \in I_2 \\ \vdots \\ \varphi_{n+1}(x,\mu), x \in I_{n+1} \end{cases}$$

of the Eq. (1.1) satisfying the conditions $\varphi^{[r-1]}(c_0+,\mu) = \xi_r$, $r = \overline{1,4}$, $\xi_r \in \mathbb{C}$. It is known that $\varphi_1^{[r-1]}(x,\mu)$ are entire functions of μ of order $1/4$ on I_1 (see [15]). Transmission conditions $B_s^r(\varphi) = 0$, $r = \overline{1,4}$, $s = \overline{1,n}$, give that all $\varphi_l^{[r-1]}(x,\mu)$, $l = \overline{2,n+1}$, are entire functions of μ of order $1/4$ on I_k, $k = \overline{1,n+1}$, except the singular point c_{n+1}. This implies that $[\varphi(x,\mu), z_m(x)](a)$, $[\varphi(x,\mu), u_m(x)](a)$, where $c_n \leq a < c_{n+1}$, have the same property.

Now consider the association $y \longleftrightarrow Y$ given in (3.1). Using the second eqution given in (3.5) on I_{n+1} in the equation (3.4) we find that

$$\left(\prod_{r=1}^{n}\Upsilon_{(r)}\right)\begin{bmatrix} u_{1(n+1)}^{[0]} & u_{2(n+1)}^{[0]} & z_{1(n+1)}^{[0]} & z_{2(n+1)}^{[0]} \\ u_{1(n+1)}^{[1]} & u_{2(n+1)}^{[1]} & z_{1(n+1)}^{[1]} & z_{2(n+1)}^{[1]} \\ u_{1(n+1)}^{[3]} & u_{2(n+1)}^{[3]} & z_{1(n+1)}^{[3]} & z_{2(n+1)}^{[3]} \\ u_{1(n+1)}^{[2]} & u_{2(n+1)}^{[2]} & z_{1(n+1)}^{[2]} & z_{2(n+1)}^{[2]} \end{bmatrix}\begin{bmatrix} [y, z_{1(n+1)}] \\ [y, z_{2(n+1)}] \\ -[y, u_{1(n+1)}] \\ -[y, u_{2(n+1)}] \end{bmatrix} = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{bmatrix}.$$

Hence y has the representation on I_{n+1}

$$\begin{aligned} y = \left(\prod_{r=1}^{n}\Upsilon_{(r)}\right)\big\{&[y, z_{1(n+1)}]u_{1(n+1)} + [y, z_{2(n+1)}]u_{2(n+1)} \\ &-[y, u_{1(n+1)}]z_{1(n+1)} - [y, u_{2(n+1)}]z_{2(n+1)}\big\}. \end{aligned} \tag{3.11}$$

Beside this, with a direct calculation we get on I_{n+1} that

$$\begin{aligned} [y, z_{1(n+1)}]^{(1)} &= \mu y z_{1(n+1)}, \\ [y, z_{2(n+1)}]^{(1)} &= \mu y z_{2(n+1)}, \\ [y, u_{1(n+1)}]^{(1)} &= \mu y u_{1(n+1)}, \\ [y, u_{2(n+1)}]^{(1)} &= \mu y u_{2(n+1)}, \end{aligned} \tag{3.12}$$

where $[,]^{(1)}$ denotes the ordinary derivative of $[,]$ with respect to the variable x. Therefore substituting (3.11) in (3.12) we get on I_{n+1} that

$$\Psi^{(1)}(x, \mu) = \mu D(x)\Psi(x, \mu), \tag{3.13}$$

where

$$\Psi(x, \mu) = \begin{bmatrix} [y, z_{1(n+1)}](x, \mu) \\ [y, z_{2(n+1)}](x, \mu) \\ [y, u_{1(n+1)}](x, \mu) \\ [y, u_{2(n+1)}](x, \mu) \end{bmatrix}$$

and

$$D(x) = \left(\prod_{r=1}^{n}\Upsilon_{(r)}\right) \times \begin{bmatrix} u_{1(n+1)}(x)z_{1(n+1)}(x) & u_{2(n+1)}(x)z_{1(n+1)}(x) & -z_{1(n+1)}^{2}(x) & -z_{1(n+1)}(x)z_{2(n+1)}(x) \\ u_{1(n+1)}(x)z_{2(n+1)}(x) & u_{2(n+1)}(x)z_{2(n+1)}(x) & -z_{1(n+1)}(x)z_{2(n+1)}(x) & -z_{2(n+1)}^{2}(x) \\ u_{1(n+1)}^{2}(x) & u_{2(n+1)}(x)u_{1(n+1)}(x) & -z_{1(n+1)}(x)u_{1(n+1)}(x) & -z_{2(n+1)}(x)u_{1(n+1)}(x) \\ u_{1(n+1)}(x)u_{2(n+1)}(x) & u_{2(n+1)}^{2}(x) & -z_{1(n+1)}(x)u_{2(n+1)}(x) & -z_{2(n+1)}(x)u_{2(n+1)}(x) \end{bmatrix}.$$

Note that the elements of $D(x)$ are integrable on I_{n+1}, since lim-4 case holds for (1.1). Integrating both side in (3.13) from a to x ($x \in I_{n+1}$) we get that

$$\Psi(x,\mu)=\Psi(a,\mu)+\mu\int_a^x D(\zeta)\Psi(\zeta,\mu)d\zeta. \tag{3.14}$$

We obtain from (3.14) that

$$\|\Psi(x,\mu)\|\leq\|\Psi(a,\mu)\|\exp\left(|\mu|\int_a^x\|D(\zeta)\|\,d\zeta\right),\ x\in I_{n+1}. \tag{3.15}$$

Using (3.15) we get on I_{n+1} that

$$\|\Psi(c_{n+1}-,\mu)-\Psi(a,\mu)\|\leq|\mu|\left(\int_a^{c_{n+1}}\|D(\zeta)\|\,d\zeta\right)\exp\left(|\mu|\int_{c_n}^{c_{n+1}}\|D(\zeta)\|\,d\zeta\right), \tag{3.16}$$

and

$$\|\Psi(c_{n+1}-,\mu)\|\leq\|\Psi(a,\mu)\|\exp\left(|\mu|\int_a^{c_{n+1}}\|D(\zeta)\|\,d\zeta\right). \tag{3.17}$$

(3.16) implies that $\Psi(a,\mu)$ converges uniformly in μ to $\Psi(c_{n+1},\mu)$ in any $\mu-$compact set as $a\to c_{n+1}-$. (3.17) implies that $\Psi(c_{n+1}-,\mu)$ is of not higher than first order. Moreover from (3.17) we arrive at $[y,z_{m(n+1)}](c_{n+1}-,\mu)$, $[y,u_{m(n+1)}](c_{n+1}-,\mu)$ are of minimal type. Now taking $y(x,\mu)$ as $\theta_m(x,\mu)$ and $\psi_m(x,\mu)$, $m=1,2$, we complete the proof. □

4 Dissipative operator

In this section we shall describe the eigenvalue problem with boundary value transmission conditions. Then we will construct an operator associated with this problem in the suitable Hilbert space.

Let us consider the following fourth order boundary value problem with finite transmission conditions

$$\varphi^{[4]}=\mu\varphi,\ \varphi\in D,\ x\in I, \tag{4.1}$$
$$B_1^-(\varphi):=\varphi^{[0]}(c_0+)-\alpha_1\varphi^{[3]}(c_0+)=0, \tag{4.2}$$
$$B_2^-(\varphi):=\varphi^{[1]}(c_0+)-\alpha_2\varphi^{[2]}(c_0+)=0, \tag{4.3}$$
$$B_s^r(y):=\varphi^{[r-1]}(c_s-)-\gamma_{rs}\varphi^{[r-1]}(c_s+)=0, \tag{4.4}$$
$$B_1^+(y):=[\varphi,z_1](c_{n+1}-)+k_1[\varphi,u_1](c_{n+1}-)=0, \tag{4.5}$$
$$B_2^+(y):=[\varphi,z_2](c_{n+1}-)+k_2[\varphi,u_2](c_{n+1}-)=0, \tag{4.6}$$

where $r=\overline{1,4}$, $s=\overline{1,n}$, $\alpha_1,\alpha_2,\gamma_{rs}$ are real numbers as given in the Sect. 2, k_1 and k_2 are complex numbers such that $k_1=\Re k_1+i\Im k_1$ and $k_2=\Re k_2+i\Im k_2$ with $\Im k_1,\Im k_2>0$.

Let $\mathcal{H} = \bigoplus_{k=1}^{n+1} \mathcal{H}_k$, where $\mathcal{H}_k = L^2(I_k)$, be the Hilbert space with the inner product

$$\langle \varphi, \chi \rangle_{\mathcal{H}} = \left[(\varphi_1, \chi_1)_{\mathcal{H}_1} \; (\varphi_2, \chi_2)_{\mathcal{H}_2} \cdots (\varphi_{n+1}, \chi_{n+1})_{\mathcal{H}_{n+1}} \right] \times \left[1 \; \Upsilon_{(1)} \cdots \prod_{r=1}^{n} \Upsilon_{(r)} \right]^t .$$

Here $\mathcal{H}_k$ are the Hilbert spaces with the usual inner products on I_k and $[.]^t$ denotes the transpose of the matrix [.].

Consider the following set

$$D(L) = \left\{ \varphi \in \mathcal{H} : \varphi^{[r-1]} \in AC_{loc}(I_k), \begin{array}{l} B_m^-(\varphi) = 0, \\ B_s^r(\varphi) = 0, \; \varphi^{[4]} \in \mathcal{H} \\ B_m^+(\varphi) = 0, \end{array} \right\},$$

where $r = \overline{1,4}$, $m = 1, 2$, $s = \overline{1,n}$, and $AC_{loc}(I_k)$ denotes the set consisting of all locally absolutely continuous functions on I_k, $k = \overline{1, n+1}$. We define the operator L on $D(L)$ as

$$L\varphi = \varphi^{[4]}, \; x \in I.$$

Hence the BVTP (4.1)–(4.6) can be introduced by the operator L in $\mathcal{H}$ as

$$L\varphi = \mu\varphi, \; \varphi \in D(L), \; x \in I.$$

A direct consequence is the following theorem.

Theorem 4.1 *L is dissipative in $\mathcal{H}$.*

Proof Consider an arbitrary element φ in $D(L)$. Then a direct calculation gives that

$$\langle L\varphi, \varphi \rangle_{\mathcal{H}} - \langle \varphi, L\varphi \rangle_{\mathcal{H}} = [\varphi, \overline{\varphi}]_{c_0+}^{c_1-} + \Upsilon_{(1)} [\varphi, \overline{\varphi}]_{c_1+}^{c_2-} + \ldots + \prod_{r=1}^{n} \Upsilon_{(r)} [\varphi, \overline{\varphi}]_{c_n+}^{c_{n+1}-}. \quad (4.7)$$

From the conditions $B_1^-(\varphi) = 0$ and $B_2^-(\varphi) = 0$ we get that

$$[\varphi, \overline{\varphi}](c_0+) = 0. \quad (4.8)$$

Further since φ satisfies the conditions $B_s^r(\varphi) = 0$, $r = \overline{1,4}$, $s = \overline{1,n}$, we have

$$[\varphi, \overline{\varphi}](c_s-) = \Upsilon_{(s)} [\varphi, \overline{\varphi}](c_s+), \; s = \overline{1,n}. \quad (4.9)$$

On the other hand from the conditions $B_1^+(\varphi) = 0$, $B_2^+(\varphi) = 0$ and (3.8) we obtain

$$[\varphi, \overline{\varphi}](c_{n+1}-) = \left(\prod_{r=1}^{n} \Upsilon_{(r)} \right) 2i \Im \left\{ k_1 \left| [\varphi, u_1](c_{n+1}-) \right|^2 + k_2 \left| [\varphi, u_2](c_{n+1}-) \right|^2 \right\}. \quad (4.10)$$

Taking into account the conditions (4.7)–(4.10) one finds that

$$\Im \langle L\varphi, \varphi \rangle_{\mathcal{H}} = \left(\prod_{r=1}^{n} \Upsilon_{(r)} \right)^2 \Im \left\{ k_1 \left| [\varphi, u_1](c_{n+1}-) \right|^2 + k_2 \left| [\varphi, u_2](c_{n+1}-) \right|^2 \right\} \tag{4.11}$$

and the proof is completed. □

From Theorem 4.1 we obtain that all eigenvalues of L lie in the closed upper half-plane.

Theorem 4.2 *L has no real eigenvalue.*

Proof Assume the contrary and let μ_0 be an eigenvalue of L. Further let $\theta_1(x, \mu_0)$ be the eigenfunction of L associated with the real eigenvalue μ_0. For the solution $\psi_1(x, \mu_0)$ of (4.1) we know that

$$[\theta_1, \psi_1](c_0+) = 1. \tag{4.12}$$

On the other side the equality

$$\Im \langle L\theta_1, \theta_1 \rangle_{\mathcal{H}} = \Im \left(\mu_0 \left\| \theta_1 \right\|_{\mathcal{H}}^2 \right) \tag{4.13}$$

holds. Therefore from (4.11) and (4.13) we arrive at

$$[\theta_1, u_1](c_{n+1}-) = [\theta_1, u_2](c_{n+1}-) = 0. \tag{4.14}$$

Using (4.14) in the conditions $B_1^+(\theta_1) = 0,\ B_2^+(\theta_1) = 0$ we obtain that

$$[\theta_1, z_1](c_{n+1}-) = [\theta_1, z_2](c_{n+1}-) = 0. \tag{4.15}$$

Taking into account the constant of the bilinear concomitant of $\theta_1(x, \mu_0)$ and $\psi_1(x, \mu_0)$ on each interval I_k, transmission conditions, (4.14) and (4.15) we get that

$$\begin{aligned} &[\theta_1, \psi_1](c_0+) = \left(\prod_{r=1}^{n} \Upsilon_{(r)} \right) [\theta_1, \psi_1](c_{n+1}-) \\ &= \left(\prod_{r=1}^{n} \Upsilon_{(r)} \right)^2 \{ [\theta_1, u_1](c_{n+1}-)[\psi_1, z_1](c_{n+1}-) - [\theta_1, z_1](c_{n+1}-)[\psi_1, u_1](c_{n+1}-) \\ &+ [\theta_1, u_2](c_{n+1}-)[\psi_1, z_2](c_{n+1}-) - [\theta_1, z_2](c_{n+1}-)[\psi_1, u_2](c_{n+1}-) \} = 0. \end{aligned} \tag{4.16}$$

However, (4.16) contradicts with (4.12). □

Therefore all eigenvalues of L lie in the open upper half-plane. In particular, zero is not an eigenvalue of L.

Now consider the function

$$\Delta(\mu) = \det \left[\mathcal{A} \Delta_1(c_0+, \mu) + \mathcal{B} \Delta_2(c_{n+1}-, \mu) \right], \tag{4.17}$$

where

$$\Delta_1(x,\mu)=\begin{bmatrix}\theta_1^{[0]}(x,\mu) & \theta_2^{[0]}(x,\mu) & \psi_1^{[0]}(x,\mu) & \psi_2^{[0]}(x,\mu)\\ \theta_1^{[1]}(x,\mu) & \theta_2^{[1]}(x,\mu) & \psi_1^{[1]}(x,\mu) & \psi_2^{[1]}(x,\mu)\\ \theta_1^{[2]}(x,\mu) & \theta_2^{[2]}(x,\mu) & \psi_1^{[2]}(x,\mu) & \psi_2^{[2]}(x,\mu)\\ \theta_1^{[3]}(x,\mu) & \theta_2^{[3]}(x,\mu) & \psi_1^{[3]}(x,\mu) & \psi_2^{[3]}(x,\mu)\end{bmatrix}, x\in I_1,$$

$$\Delta_2(x,\mu)=\begin{bmatrix}[\theta_1,z_1](x,\mu) & [\theta_2,z_1](x,\mu) & [\psi_1,z_1](x,\mu) & [\psi_2,z_1](x,\mu)\\ [\theta_1,z_2](x,\mu) & [\theta_2,z_2](x,\mu) & [\psi_1,z_2](x,\mu) & [\psi_2,z_2](x,\mu)\\ [\theta_1,u_1](x,\mu) & [\theta_2,u_1](x,\mu) & [\psi_1,u_1](x,\mu) & [\psi_2,u_1](x,\mu)\\ [\theta_1,u_2](x,\mu) & [\theta_2,u_2](x,\mu) & [\psi_1,u_2](x,\mu) & [\psi_2,u_2](x,\mu)\end{bmatrix}, x\in I_{n+1}.$$

and

$$\mathcal{A}=\begin{bmatrix}1 & 0 & 0 & -\alpha_1\\ 0 & 1 & -\alpha_2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\ \mathcal{B}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & k_1 & 0\\ 0 & 1 & 0 & k_2\end{bmatrix}.$$

$\Delta(\mu)$ is called the characteristic function of L and it is known that the zeros of $\Delta(\mu)$ coincide with the eigenvalues of L (see [14]). Using Theorem 3.1 we obtain that $\Delta(\mu)$ is an entire function of μ of order≤ 1 and minimal type.

Following the same idea constructed in (4.17) one may obtain that the zeros of the function

$$\Delta_{(\Re)}(\mu)=\det\begin{bmatrix}[\theta_1,z_1](c_{n+1}-)+\Re k_1[\theta_1,u_1](c_{n+1}-) & [\theta_2,z_1](c_{n+1}-)+\Re k_1[\theta_2,u_1](c_{n+1}-)\\ [\theta_1,z_2](c_{n+1}-)+\Re k_2[\theta_1,u_2](c_{n+1}-) & [\theta_2,z_2](c_{n+1}-)+\Re k_2[\theta_2,u_2](c_{n+1}-)\end{bmatrix} \tag{4.18}$$

coincides with the eigenvalues of the real part L_1 of the operator L. Further we can infer that $\Delta_{(\Re)}(\mu)$ is an entire function of μ of order≤ 1 and minimal type.

5 Resolvent operator and Krein's theorem

In this section we describe the inverse operator of L. Then we use Krein's theorem to get the complete spectral information for the operator L.

Let us consider the equation

$$L\varphi=a(x), \tag{5.1}$$

where $\varphi\in D(L),\ x\in I,\ a\in\mathcal{H}$ such that

$$a(x)=\begin{cases}a_1(x), & x\in I_1\\ a_2(x), & x\in I_2\\ \vdots & \\ a_{n+1}(x), & x\in I_{n+1}\end{cases}.$$

One can handle the equation (5.1) with an equivalent Hamiltonian system

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi^{[0]} \\ \varphi^{[1]} \\ \varphi^{[3]} \\ \varphi^{[2]} \end{bmatrix}^{(1)} - \begin{bmatrix} -q_0 & 0 & 0 & 0 \\ 0 & -q_1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi^{[0]} \\ \varphi^{[1]} \\ \varphi^{[3]} \\ \varphi^{[2]} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$J\Phi^{(1)} - Q\Phi = WA \tag{5.2}$$

and equivalent boundary value transmission conditions

$$\begin{bmatrix} 1 & 0 & -\alpha_1 & 0 \\ 0 & 1 & 0 & -\alpha_2 \end{bmatrix} \Phi(c_0+) = 0, \tag{5.3}$$

$$\Phi(c_s-) = \begin{bmatrix} \gamma_{1s} & 0 & 0 & 0 \\ 0 & \gamma_{2s} & 0 & 0 \\ 0 & 0 & \gamma_{4s} & 0 \\ 0 & 0 & 0 & \gamma_{3s} \end{bmatrix} \Phi(c_s+), \tag{5.4}$$

$$\lim_{x\to c_{n+1}-} \begin{bmatrix} 1 & 0 & k_1 & 0 \\ 0 & 1 & 0 & k_2 \end{bmatrix} \mathcal{Y}^t(x) J\Phi(x) = 0, \tag{5.5}$$

where $s = \overline{1, n}$, Φ is the associated vector of φ under (3.1),

$$\Phi = \begin{cases} \Phi_1, \ x \in I_1 \\ \Phi_2, \ x \in I_2 \\ \vdots \\ \Phi_{n+1}, \ x \in I_{n+1} \end{cases}, \ \mathcal{Y} = \begin{cases} \mathcal{Y}_1, \ x \in I_1 \\ \mathcal{Y}_2, \ x \in I_2 \\ \vdots \\ \mathcal{Y}_{n+1}, \ x \in I_{n+1} \end{cases},$$

$\mathcal{Y}_k = [Z_{1k}, Z_{2k}, U_{1k}, U_{2k}]$, and Z_{mk} and U_{mk} ($m = 1, 2,\ k = \overline{1, n+1}$) are the associated vectors of z_{mk} and u_{mk}, respectively, under (3.1).

Note that

$$\mathcal{Y}_1^t J \mathcal{Y}_1 = -J,\ \mathcal{Y}_2^t J \mathcal{Y}_2 = -\frac{1}{\Upsilon_{(1)}} J,\ \cdots,\ \mathcal{Y}_{n+1}^t J \mathcal{Y}_{n+1} = -\frac{1}{\prod_{r=1}^{n} \Upsilon_{(r)}} J. \tag{5.6}$$

Using the method of variation of parameters and (5.6) one can write

$$\begin{aligned}
\Phi_1 &= \mathcal{Y}_1(x) \int_{c_0}^{x} J\mathcal{Y}_1^t W_1 A_1 d\zeta + \mathcal{Y}_1(x) D_1,\ x \in I_1, \\
\Phi_2 &= \Upsilon_{(1)} \mathcal{Y}_2(x) \int_{c_1}^{x} J\mathcal{Y}_2^t W_2 A_2 d\zeta + \mathcal{Y}_2(x) D_2,\ x \in I_2, \\
&\vdots \\
\Phi_{n+1} &= \prod_{r=1}^{n} \Upsilon_{(r)} \mathcal{Y}_{n+1}(x) \int_{c_n}^{x} J\mathcal{Y}_{n+1}^t W_{n+1} A_{n+1} d\zeta + \mathcal{Y}_{n+1}(x) D_{n+1},\ x \in I_{n+1},
\end{aligned} \tag{5.7}$$

where D_k $(k = \overline{1, n+1})$ are constants and W_k and A_k are the parts of W and A, respectively, on I_k. Using (5.7) and the conditions (5.3)-(5.5), the solution Φ of (5.2) is found as

$$\Phi(x) = \int_{I_1} \mathcal{G}(x,\zeta)W(\zeta)A(\zeta)d\zeta + \Upsilon_{(1)} \int_{I_2} \mathcal{G}(x,\zeta)W(\zeta)A(\zeta)d\zeta$$
$$+ \ldots + \prod_{r=1}^{n} \Upsilon_{(r)} \int_{I_{n+1}} \mathcal{G}(x,\zeta)W(\zeta)A(\zeta)d\zeta, \tag{5.8}$$

where

$$\mathcal{G}(x,\zeta) = \begin{cases} -\mathbf{V}(x)\mathbf{U}^t(\zeta), & c_0 \le \zeta \le x \le c_{n+1} \\ -\mathbf{U}(x)\mathbf{V}^t(\zeta), & c_0 \le x \le \zeta \le c_{n+1} \end{cases}$$

and

$$\mathbf{V}(x) = \mathcal{Y}(x)\begin{bmatrix} I \\ K \end{bmatrix}, \ \mathbf{U}(x) = [U_1, U_2], \ K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$

I is the 2×2 identity matrix.

We construct the kernel

$$G(x,\zeta) = \begin{cases} -\mathcal{V}^t(\zeta)\mathcal{U}(x), & c_0 \le x \le \zeta \le c_{n+1}, \ x,\zeta \ne c_m, \ m = \overline{1,n} \\ -\mathcal{V}^t(x)\mathcal{U}(\zeta), & c_0 \le \zeta \le x \le c_{n+1}, \ x,\zeta \ne c_m, \ m = \overline{1,n} \end{cases}, \tag{5.9}$$

where $m = \overline{1,n}$,

$$\mathcal{U}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \ \mathcal{Z}(x) = \begin{bmatrix} z_1(x) \\ z_2(x) \end{bmatrix}, \ K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$
$$\mathcal{V}(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} = \mathcal{Z}(x) + K\mathcal{U}(x)$$

and

$$\mathcal{U}(x) = \begin{cases} \mathcal{U}_1(x), x \in I_1 \\ \mathcal{U}_2(x), x \in I_2 \\ \vdots \\ \mathcal{U}_{n+1}(x), x \in I_{n+1} \end{cases}, \ \mathcal{V}(x) = \begin{cases} \mathcal{V}_1(x), x \in I_1 \\ \mathcal{V}_2(x), x \in I_2 \\ \vdots \\ \mathcal{V}_{n+1}(x), x \in I_{n+1} \end{cases}.$$

Therefore from (5.8) the solution φ of (5.1) is obtained as

$$\varphi(x) = \int_{I_1} G(x,\zeta)a(\zeta)d\zeta + \Upsilon_{(1)} \int_{I_2} G(x,\zeta)a(\zeta)d\zeta + \ldots + \prod_{r=1}^{n} \Upsilon_{(r)} \int_{I_{n+1}} G(x,\zeta)a(\zeta)d\zeta$$

or

$$\varphi(x) = \langle G(x,\zeta), \overline{a}(\zeta) \rangle_{\mathcal{H}}.$$

For arbitrary $a \in \mathcal{H}$ we define the operator T as follows

$$Ta = \langle G(x,\zeta), \overline{a}(\zeta)\rangle_{\mathcal{H}}. \tag{5.10}$$

It is clear that T is the inverse operator of L and is a Hilbert–Schmidt operator in the Hilbert space $\mathcal{H}$. This implies that the root vectors of the operators T and L coincide. Therefore the completeness of the system of all eigen- and associated vectors of T is equivalent to the completeness of those for L. Now we shall remind Krein's theorem.

Krein's Theorem *([28], p. 238) Let K be a compact dissipative operator in H with nuclear imaginary part $\Im K$. The system of all root vectors of K is complete in H so long as at least one of the following two conditions is fulfilled:*

$$\lim_{r\to\infty}\frac{n_+(r,\Re K)}{r}=0,\ \lim_{r\to\infty}\frac{n_-(r,\Re K)}{r}=0,$$

where $n_+(r,\Re K)$ and $n_-(r,\Re K)$ denote the number of characteristic values of the real component $\Re K$ of K in the intervals $[0,r]$ and $[-r,0]$, respectively.

Following theorem will help us to use Krein's theorem.

Theorem 5.1 *[27] If an entire function $h(\mu)$ is of order ≤ 1 and minimal type, then*

$$\lim_{\rho\to\infty}\frac{n_+(\rho,h)}{\rho}=\lim_{\rho\to\infty}\frac{n_-(\rho,h)}{\rho}=0,$$

where $n_+(\rho,h)$ and $n_-(\rho,h)$ denote the number of the zeros of the function $h(\mu)$ in the intervals $[0,\rho]$ and $[-\rho,0]$, respectively.

We can write the operator T as the sum of two operators. In fact, since $k_m = \Re k_m + i\Im k_m$, $m = 1, 2$, one can derive from (5.9) and (5.10) that $T = T_1 + iT_2$, where

$$T_1 a = \langle G_1(x,\zeta), \overline{a}(\zeta)\rangle_{\mathcal{H}},\ T_2 a = \langle G_2(x,\zeta), \overline{a}(\zeta)\rangle_{\mathcal{H}},$$

and

$$G_1(x,\zeta)=\begin{cases} -[\mathcal{Z}(\zeta)+\Re K\mathcal{U}(\zeta)]^t\,\mathcal{U}(x), & c_0\leq x\leq\zeta\leq c_{n+1},\ x,\zeta\neq c_m,\ m=\overline{1,n} \\ -[\mathcal{Z}(x)+\Re K\mathcal{U}(x)]^t\,\mathcal{U}(\zeta), & c_0\leq\zeta\leq x\leq c_{n+1},\ x,\zeta\neq c_m,\ m=\overline{1,n}\end{cases},$$

$$G_2(x,\zeta) = -[\Im K\mathcal{U}(\zeta)]^t\,\mathcal{U}(x).$$

T_1 is a selfadjoint Hilbert–Schmidt operator and T_2 is a selfadjoint rank-two operator. Moreover a direct calculation shows that $\langle T_2 a, a\rangle_{\mathcal{H}} \leq 0$. It is not so hard to see that T_1 is the inverse of the real part L_1 of the operator L.

Consider the operator $-T$, $-T = -T_1 - iT_2$. Note that $-T$ is dissipative in $\mathcal{H}$. The characteristic values of the operator $-T_1$ coincide with the eigenvalues of the operator L_1. Therefore using (4.18), Theorem 5.1 and Krein's Theorem we arrive at the following results.

Theorem 5.2 *All root vectors of the operator $-T$ (also T) span the Hilbert space $\mathcal{H}$.*

Therefore from all the obtained results throughout the paper we can introduce the following theorem.

Theorem 5.3 *The spectrum of the BVTP (4.1)–(4.6) consist of purely discrete eigenvalues with finite multiplicity and belong to the open upper half-plane. The system of all eigen- and associated functions of the BVTP (4.1)–(4.6) span the Hilbert space $\mathcal{H}$.*

6 Conclusion

In this paper we have considered a fourth order differential equation defined on multi-interval in lim-4 case subject to the boundary value transmission conditions. To investigate this problem we have obtained some identities and results about the growth of the entire functions. Moreover we have described a suitable Hilbert space with a special inner product. Then we have proved that corresponding operator associated with the problem is dissipative in this Hilbert space. To get the complete spectral properties of this dissipative operator we have used Krein's theorem. Hence we have passed to the inverse operator with explicit form. Finally we have proved that all eigen- and associated vectors of this operator (problem) span the Hilbert space.

To be more precise we shall give an example.

Let us consider the following fourth order differential equation

$$\varphi^{(4)} - a(x^{\alpha}\varphi^{(1)})^{(1)} + bx^{\beta}\varphi = \mu\varphi,\ \ I \subset (1, \infty). \tag{6.1}$$

Devinatz [21] proved that for $\beta = 2\alpha,\ \alpha > \frac{2}{3}$ and $a \pm (a^2 - 4b)^{\frac{1}{2}} < 0$, (6.1) has four linearly independent solutions belonging to $L^2(I)$. Then we arrive at all eigenvalues of the problem (6.1), (4.2)–(4.6) belong to the open upper halp-plane and they are purely discrete. Further all eigen- and associated functions of this problem span the Hilbert space.

References

1. R.C. Rach, J.S. Duan, A.M. Wazwaz, Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. J. Math. Chem. **52**(1), 255–267 (2014)
2. P.M. Lima, L. Morgado, Numerical modeling of oxygen diffusion in cells with Michaelis–Menten uptake kinetics. J. Math. Chem. **48**(1), 145–158 (2010)
3. R. Singh, G. Nelakanti, J. Kumar, A new efficient technique for solving two-point boundary value problems for integro-differential equations. J. Math. Chem. **52**, 2030–2051 (2014)
4. R. Singh, J. Kumar, G. Nelakanti, Approximate series solution of fourth-order boundary value problems using decomposition method with Green's function. J. Math. Chem. **52**, 1099–1118 (2014)
5. J. Vigo-Aguiar, Applied differential equations and related computational mathematics in chemistry. J. Math. Chem. **52**, 1021–1022 (2014)
6. A. Davies, A. Karageorghis, T. Phillips, Spectral Galerkin methods for the primary two-point boundary value problem in modelling viscoelastic flows. Int. J. Numer. Meth. Eng. **26**(3), 647–662 (1988)
7. T.F. Ma, J. da Silva, Iterative solutions for a beam equation with nonlinear boundary conditions of third order. Appl. Math. Comput. **159**(1), 11–18 (2004)
8. M. Chawla, C. Katti, Finite difference methods for two-point boundary value problems involving high order differential equations. BIT Numer. Math. **19**(1), 27–33 (1979)

9. J.R. Graef, L. Kong, Q. Kong, B. Yang, Positive solutions to a fourth order boundary value problem. Results Math. **59**(1–2), 141–155 (2011)
10. V. Lakshmikantham, D.D. Bainov, P.S. Simenov, *Theory of impulsive differential equations, series in modern applied mathematics*, vol. 6 (World Scientific Publishing Co. Inc, Teaneck, 1989)
11. Z. Wang, H. Wu, Dissipative non-self-adjoint Sturm–Liouville operators and completeness of their eigenfunctions. J. Math. Anal. Appl. **394**, 1–12 (2012)
12. G. Guseinov, Completeness theorem for the dissipative Sturm–Liouville operator. Doga-Tr. J. Math. **17**, 48–54 (1993)
13. E. Bairamov, E. Ugurlu, Krein's theorems for a dissipative boundary value transmission problems. Complex Anal. Oper. Theory **7**, 831–842 (2013)
14. X.-Y. Zhang, J. Sun, The determinants of fourth order dissipative operators with transmission conditions. J. Math. Anal. Appl. **410**, 55–69 (2014)
15. M.A. Naimark, *Linear differential operators*, 2nd edn., (Nauka, Moscow, 1969); English transl. of 1st edn, Parts 1, 2, Ungar, New York, 1967, 1968.
16. W.N. Everitt, The Sturm–Liouville problem for fourth order differential equations. Q. J. Math. **8**, 146–160 (1957)
17. W.N. Everitt, Fourth order singular differential equations. Math. Ann. **149**, 320–340 (1963)
18. H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen wilkürlichen Funktionen. Math. Ann. **68**, 220–269 (1910)
19. E.C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations, Part 1*, 2nd edn. (Oxford University Press, Oxford, 1962)
20. M.S.P. Eastham, The lim-$2n$ case of symmetric differential operators of order $2n$. Proc. Lond. Math. Soc. **3**(38), 272–294 (1979)
21. A. Devinatz, The deficiency index of certain fourth-order ordinary self-adjoint differential operators. Q. J. Math. Ser. **23**, 267–286 (1972)
22. P.W. Walker, Asymptotics for a class of fourth order differential equations. J. Diff. Eq. **11**, 321–334 (1972)
23. M.S.P. Eastham, The limit-4 case of fourth-order self-adjoint differential equations. Proc. Roy. Soc. Edinb. **79A**, 51–59 (1977)
24. R.I. Becker, Limit circle criteria for fourth order differential operators with an oscillatory coefficient. Proc. Edinb. Math. Soc. **24**, 105–117 (1981)
25. C.T. Fulton, The Bessel-Squared equation in the lim-2, lim-3 and lim-4 cases. Q. J. Math. **2**(40), 423–456 (1989)
26. BJa Levin, *Distribution of zeros of entire functions* (American Mathematical Society, Providence, 1964)
27. M.G. Krein, On the indeterminate case of the Sturm–Liouville boundary problem in the interval $(0, \infty)$. Izv. Akad. Nauk SSSR Ser. Mat. **16**(4), 293–324 (1952). (in Russian)
28. I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear non-self-adjoint operators*, in Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, RI, 1969.